Parametric Kalman filter : toward an alternative to the EnKF?

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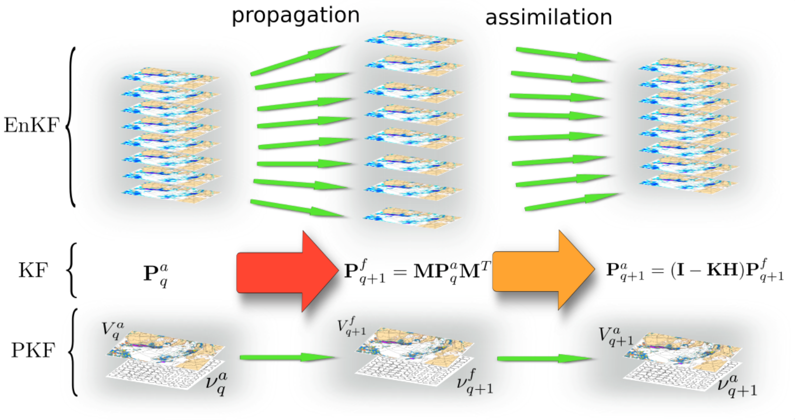
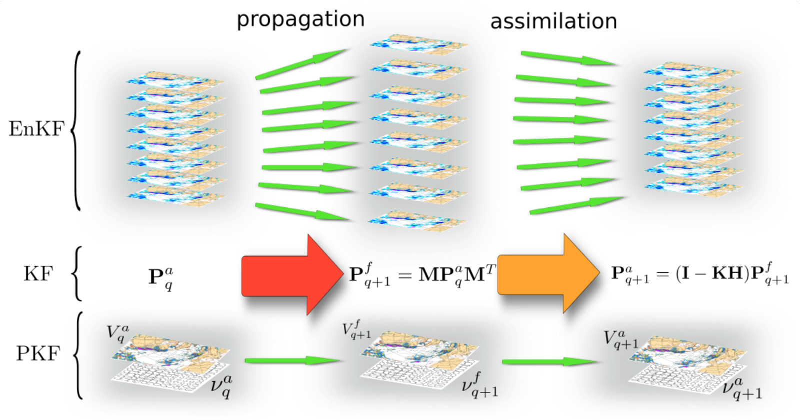
Adjoint Workshop on Sensitivity Analysis and Data Assimilation in Meteorology and Oceanography 1-6 July 2018, Aveiro, Portugal.

Kalman filter



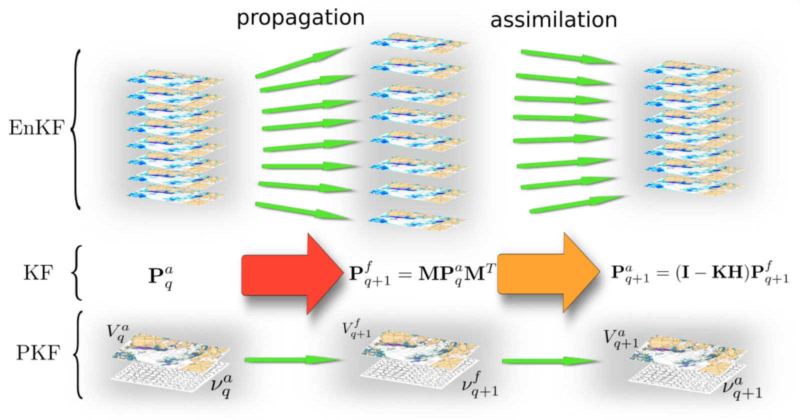
KF is a simple algorithm, but numerically costly, and not specific to our equations.

Ensemble Kalman filter



EnKF is a robust algorithm with a naturel parallel implementation, but computation is often made at a lower resolution, it is suffring from samplin noise (localization, imperfect balances,..) it is not specific to the equations of the flow.

Parametric Kalman Filter



What are the PKF equations for the analysis and the forecast steps ?

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Principle of Parametric Kalman Filter

PKF basic idea:

approximating covariances by anisotropic covariance model
(note that EnKF approximates covariances by ensemble estimation)

Parametric Kalman Filter

1. Consider a parametric covariance model,
2. Write parameter dynamics along analysis and forecast cycles.

Example of interesting parameters:

the variance and the anisotropy

PKF: diffusion based covariance model

The covariance model based on the diffusion equation [Weaver and Courtier, 2001] (mainly) writes

$$\mathbf{P} = \mathbf{\Sigma}\mathbf{L}\mathbf{L}^T\mathbf{\Sigma}^T, \tag{1}$$

where $\mathbf{\Sigma}$ stands for the diagonal matrix of grid-points standard-deviation and where

$$\mathbf{L} = e^{\mathcal{L}\frac{1}{2}}, \text{ with } \mathcal{L}(u) = \nabla \cdot (\boldsymbol{\nu}\nabla u), \tag{2}$$

is the propagator of the diffusion equation $\partial_\tau u = \mathcal{L}(u)$ from $\tau = 0$ to $1/2$.

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$$\boldsymbol{\nu}_{\boldsymbol{x}} = \frac{1}{2} \boldsymbol{g}_{\boldsymbol{x}}{}^{-1} \tag{3}$$

where $\boldsymbol{g}_{\mathbf{x}}$ features the correlation function anosotropy at $\mathbf{x}$ and is defined from

$$\rho(\mathbf{x}, \mathbf{x} + \delta\mathbf{x}) = 1 - \frac{1}{2} ||\delta\mathbf{x}||^2_{\boldsymbol{g}_{\mathbf{x}}} + \mathcal{O}(||\delta\mathbf{x}||^3). \tag{4}$$

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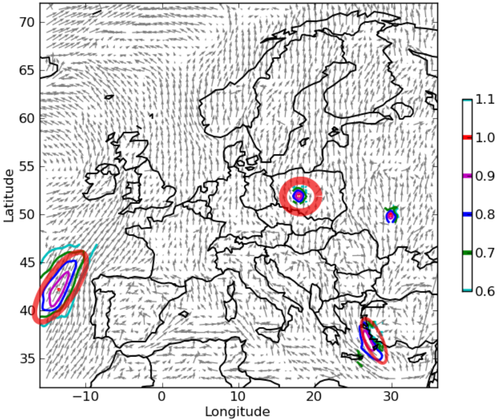
Variance and metric fields are the parameters of the covariance model based on the diffusion equation.

Shape of local correlaton functions

$$\rho(\mathbf{x}, \mathbf{x} + \delta\mathbf{x}) = 1 - \frac{1}{2}||\delta\mathbf{x}||^2_{\boldsymbol{g}_\mathbf{x}} + \mathcal{O}(||\delta\mathbf{x}||^3), \quad (5)$$

the metric $\boldsymbol{g}_\mathbf{x}$ features the shape of the local correlation function at $\mathbf{x}$



Mean flow and Anisotropy for few correlation functions [Jaumouillé et al., 2013]

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Analysis update of the parametric formulation

Require: Fields of $\boldsymbol{\nu}^b$ and $\boldsymbol{V}^b$, $\boldsymbol{V}^o$ and location $\boldsymbol{x}_j$ of the p observations to assimilate

for $j = 1 : p$ **do**

  0- Initialization of intermediate quantities

  $\boldsymbol{V}_j^b = \boldsymbol{V}_{\boldsymbol{x}_j}^b$, $\boldsymbol{V}_j^o = \boldsymbol{V}_{\boldsymbol{x}_j}^o$, $\boldsymbol{\nu}_j = \boldsymbol{\nu}_{\boldsymbol{x}_j}^b$

  $$\rho_j(\boldsymbol{x}) = \exp\left(-\frac{1}{4}||\boldsymbol{x} - \boldsymbol{x}_j||^2_{\boldsymbol{\nu}_j^{-1}}\right)$$

  1- Computation of analysis statistics

  $$\boldsymbol{V}_{\boldsymbol{x}}^a = \boldsymbol{V}_{\boldsymbol{x}}^b\left(1 - \rho_j^2(\boldsymbol{x})\frac{\boldsymbol{V}_j^b}{\boldsymbol{V}_j^b + \boldsymbol{V}_j^o}\right)$$

  $$\boldsymbol{\nu}_{\boldsymbol{x}}^a = \boldsymbol{\nu}_{\boldsymbol{x}}^b\left(1 - \rho_j^2(\boldsymbol{x})\frac{\boldsymbol{V}_j^b}{\boldsymbol{V}_j^b + \boldsymbol{V}_j^o}\right)$$

  2- Update of the background statistics

  $\boldsymbol{V}_{\boldsymbol{x}}^b \leftarrow \boldsymbol{V}_{\boldsymbol{x}}^a$

  $\boldsymbol{\nu}_{\boldsymbol{x}}^b \leftarrow \boldsymbol{\nu}_{\boldsymbol{x}}^a$

end for

Return fields $\boldsymbol{\nu}^a$ and $\boldsymbol{V}^a$

Algorithm 1: Iterated process building analysis covariance matrix at the leading order, under Gaussian shape assumption. [Pannekoucke et al., 2016]

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Parameter dynamics for linear advection/diffusion

The linear advection-diffusion dynamics writes

$$\partial_t \alpha + \boldsymbol{u} \cdot \nabla \alpha = \kappa \nabla^2 \alpha, \quad (10)$$

where u denotes the velocity and κ the diffusion rate.

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The time evolution of the variance and the diffusion tensor is given at the leading order by [Pannekoucke et al., 2016]

$$\left\{ \begin{array}{l} \partial_t \boldsymbol{\nu}^f + \boldsymbol{u}\nabla\boldsymbol{\nu}^f = \boldsymbol{\nu}^f(\nabla\boldsymbol{u})^T + (\nabla\boldsymbol{u})\boldsymbol{\nu}^f + 2\boldsymbol{\kappa}, \\ \partial_t \boldsymbol{V}^f + \boldsymbol{u}\nabla\boldsymbol{V}^f = -\boldsymbol{V}^f\, Tr\left[\left(\boldsymbol{\nu}^f\right)^{-1}\boldsymbol{\kappa}\right]. \end{array} \right. \tag{11}$$

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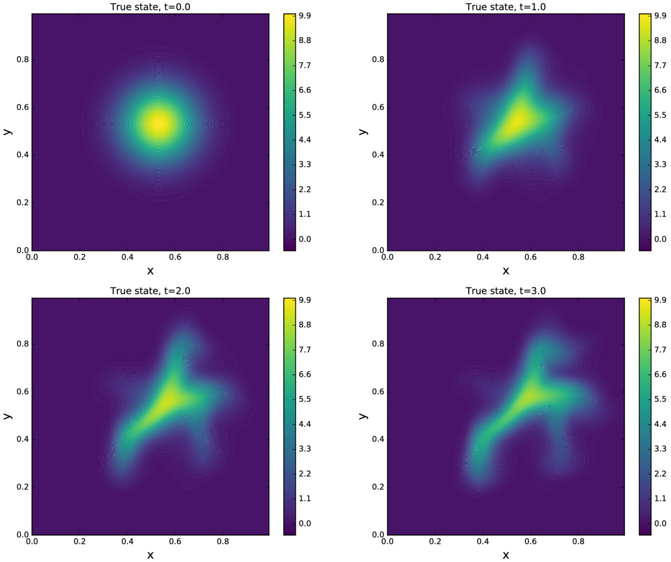
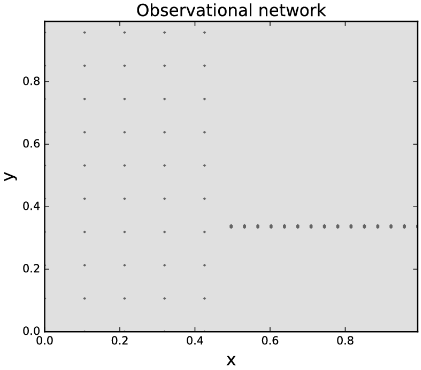
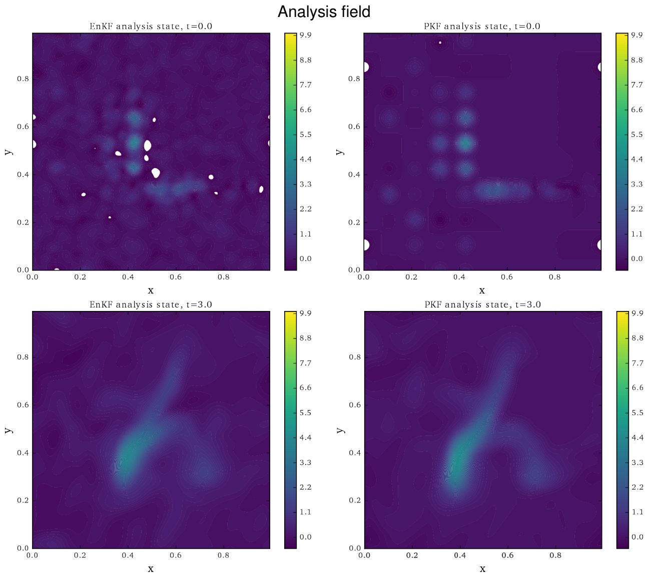
There is a coupling between the error variance and local diffusion tensor fields due to the diffusion process.



Advection-diffusion of a passive tracer



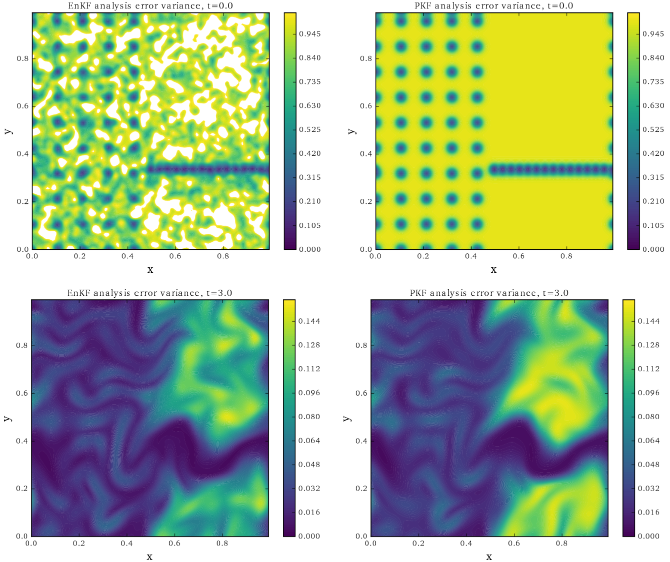
Regular network (left side) and simulated flight tracks (right side).



EnKF (EnKF == $Ne = 100$ + EnVar + Localization) vs. PKF

Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)

Variance field



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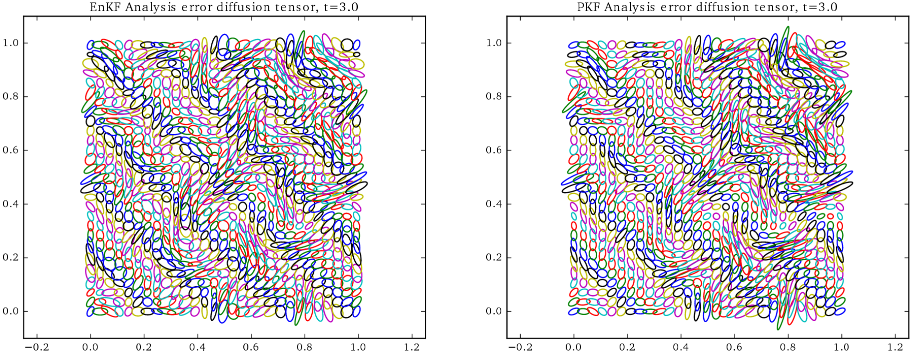
Analysis/forecast cycles, passive tracer: EnKF vs. PKF (dt=0.25)

Diffusion tensor field



EnKF (EnKF == Ne = 100 + EnVar + Localization) vs. PKF

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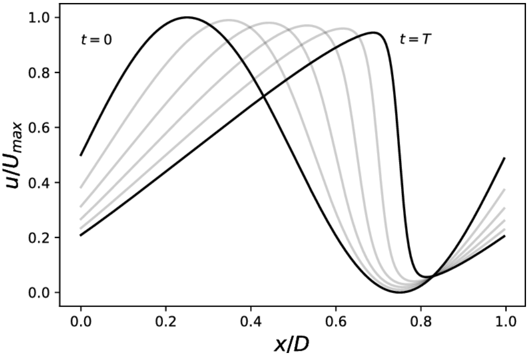
Diffusive non-linear Burgers dynamics

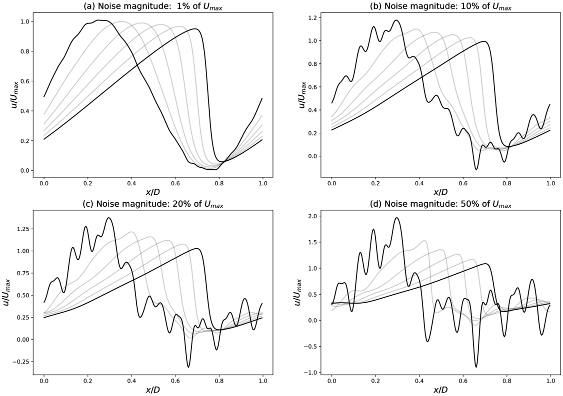
Burgers equation writes

$$\partial_t u + u\partial_x u = \kappa \partial_x^2 u. \tag{12}$$

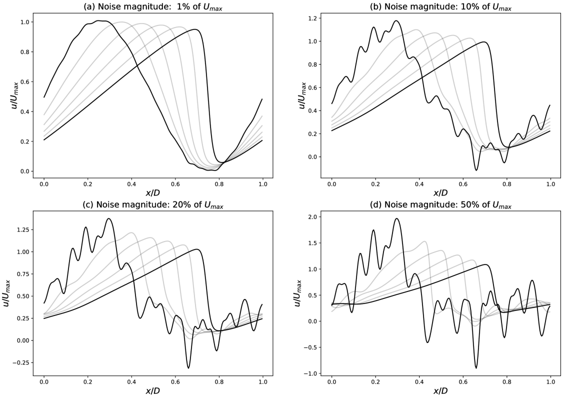


Solution starting from a cosine-like function.

Fluctuation-mean flow dynamics



Fluctuation-mean flow dynamics



With $u = \bar{u} + \varepsilon$, $\bar{\cdot} \equiv \mathbb{E}[\cdot]$ being the expectation operator, the fluctuation-mean flow dynamics for small perturbations writes :

$$\left\{ \begin{array}{l} \partial_t \bar{u} + \bar{u}\partial_x \bar{u} = \kappa \partial_x^2 \bar{u} - \overline{\varepsilon \partial_x \varepsilon}, \\ \partial_t \varepsilon + \bar{u}\partial_x \varepsilon = -\varepsilon \partial_x \bar{u} + \kappa \partial_x^2 \varepsilon. \end{array} \right. \tag{13}$$

PKF forecast dynamics for diffusive Burgers

The dynamics of the parameters writes [Pannekoucke et al., 2018]

$$\left\{\begin{array}{llll} (a) & \partial_t \bar{u} + \bar{u}\partial_x \bar{u} & = & \kappa \partial_x^2 \bar{u} - \frac{1}{2}\partial_x V, \\ (b) & \partial_t V + \bar{u}\partial_x V & = & -2(\partial_x \bar{u})V + \kappa\partial_x^2 V - \frac{\kappa}{2}\frac{1}{V}(\partial_x V)^2 - \frac{\kappa}{\nu}V, \\ (c) & \partial_t \nu + \bar{u}\partial_x \nu & = & 2(\partial_x \bar{u})\nu + 2\kappa - 2\frac{\kappa}{V}\partial_x^2 V\,\nu + 2\frac{\kappa}{V^2}(\partial_x V)^2\nu + \kappa\frac{1}{V}\partial_x V\partial_x \nu + \kappa\partial_x^2 \nu - 2\kappa\frac{1}{\nu}(\partial_x \nu)^2. \end{array}\right.$$

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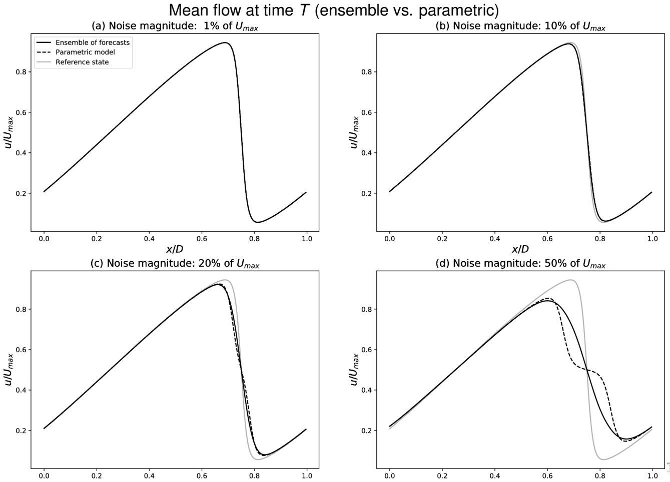

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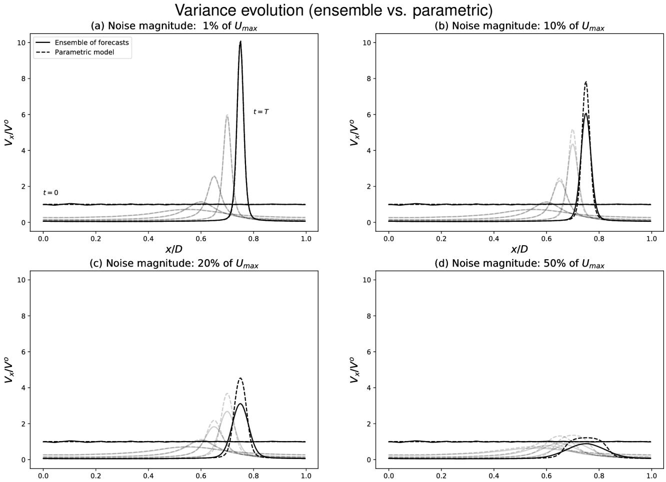
\begin{cases}
(a) & \partial_t \bar{u} + \bar{u}\partial_x \bar{u} = \kappa \partial_x^2 \bar{u} - \frac{1}{2}\partial_x V, \\
(b) & \partial_t V + \bar{u}\partial_x V = -2(\partial_x \bar{u})V + \kappa\partial_x^2 V - \frac{\kappa}{2}\frac{1}{V}(\partial_x V)^2 - \frac{\kappa}{\nu}V, \\
(c) & \partial_t \nu + \bar{u}\partial_x \nu = 2(\partial_x \bar{u})\nu + 2\kappa - 2\frac{\kappa}{V}\partial_x^2 V\,\nu + 2\frac{\kappa}{V^2}(\partial_x V)^2\nu + \kappa\frac{1}{V}\partial_x V\partial_x \nu + \kappa\partial_x^2\nu - 2\kappa\frac{1}{\nu}(\partial_x \nu)^2.
\end{cases}
$$



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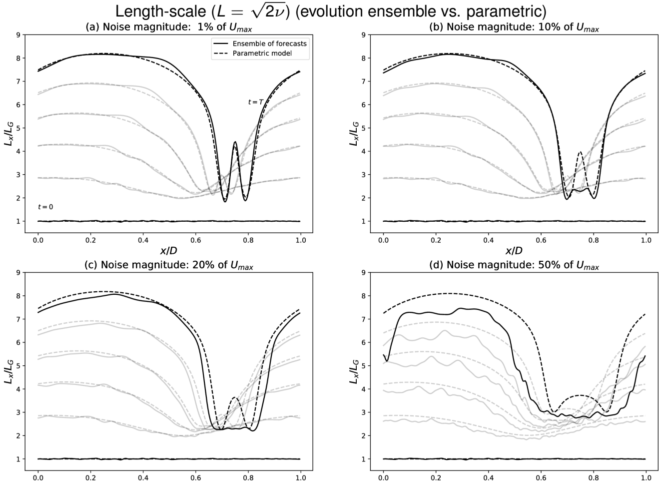

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Conclusion

The Parametric Kalman Filter relies on covariance model to reproduce the uncertainty dynamics all along analysis/forecast cycles:

- no ensemble $\Rightarrow$ no sampling noise & no localization
- low numerical cost,
- relies on a given covariance model,
- needs to describe parameters update during the analysis step,
- needs to develop dynamical equations for parameters,
- provides a new tool for understanding covariance dynamics for partial differential dynamics along analysis/forecast cycles

Further directions:

- applications for chemical transport model,
- extension for ocean/atmosphere dynamics.

Jaumouillé, E., Emili, E., Pannekoucke, O., Massart, M., and Piacentini, A. (2013).
Modelisation dynamique de la matrice des covariances d'erreur d'ebauche avec valentina-ensemble.
ACHILLE Newsletter, 11:4–8.

Pannekoucke, O., Bocquet, M., and Ménard, R. (2018).
Parametric covariance dynamics for the nonlinear diffusive burgers' equation.
Nonlinear Processes in Geophysics Discussions, 2018:1–21.

Pannekoucke, O. and Massart, S. (2008).
Estimation of the local diffusion tensor and normalization for heterogeneous correlation modelling using a diffusion equation.
Q. J. R. Meteorol. Soc., 134:1425–1438.

Pannekoucke, O., Ricci, S., Barthelemy, S., Menard, R., and Thual, O. (2016).
Parametric kalman filter for chemical transport model.
Tellus, 68:31547.

Weaver, A. and Courtier, P. (2001).
Correlation modelling on the sphere using a generalized diffusion equation (tech. memo. ecmwf, num. 306).
Quarterly Journal of the Royal Meteorological Society, 127:1815–1846.