# How to use 'fast' observations in 'slow' models?

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National Centre for Earth Observation
NATURAL ENVIRONMENT RESEARCH COUNCIL
Data Assimilation Research Centre
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# Outline

1. Time scales in nature, forecast and observations.
2. DA example
3. What do we give the DA system?

# DA problem setup

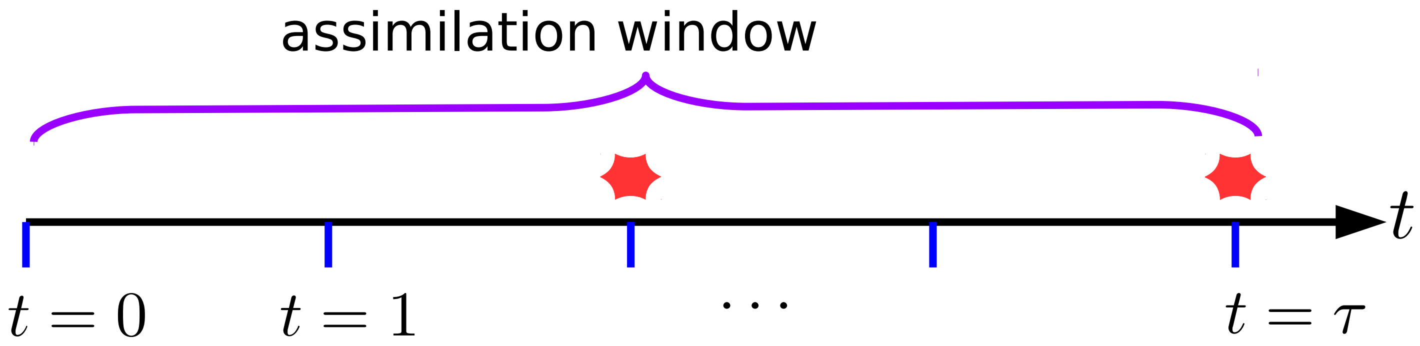
$\mathbf{x}^t \in \mathcal{R}^{N_x}$ **Model variables**

$\mathbf{y}^l \in \mathcal{R}^{N_y}$ **Observations**

$$\mathbf{x}^t = m^{(t-1)\to t}\left(\mathbf{x}^{t-1}\right) + \mathbf{v}^t$$

$$\mathbf{y}^l = h^l\left(\mathbf{x}^{t=l}\right) + \boldsymbol{\eta}^l$$

$$\{\mathbf{x}^0, \mathbf{v}^t, \boldsymbol{\eta}^l\}\ r.v.,\ \mathbf{x}^0 \perp \mathbf{v}^t \perp \boldsymbol{\eta}^l$$



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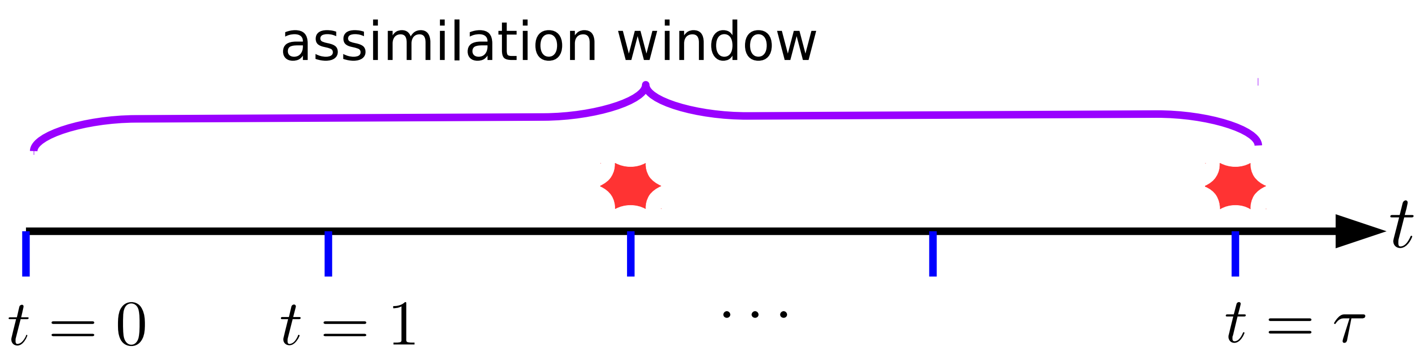


To obtain the **posterior** pdf we can use **Bayes' theorem**.

$$p\left(\mathbf{x}^{0:\tau}|\mathbf{y}^{1,L}\right) = \frac{p\left(\mathbf{y}^{1:L}|\mathbf{x}^{0:\tau}\right) p\left(\mathbf{x}^{0:\tau}\right)}{p\left(\mathbf{y}^{1:L}\right)}$$

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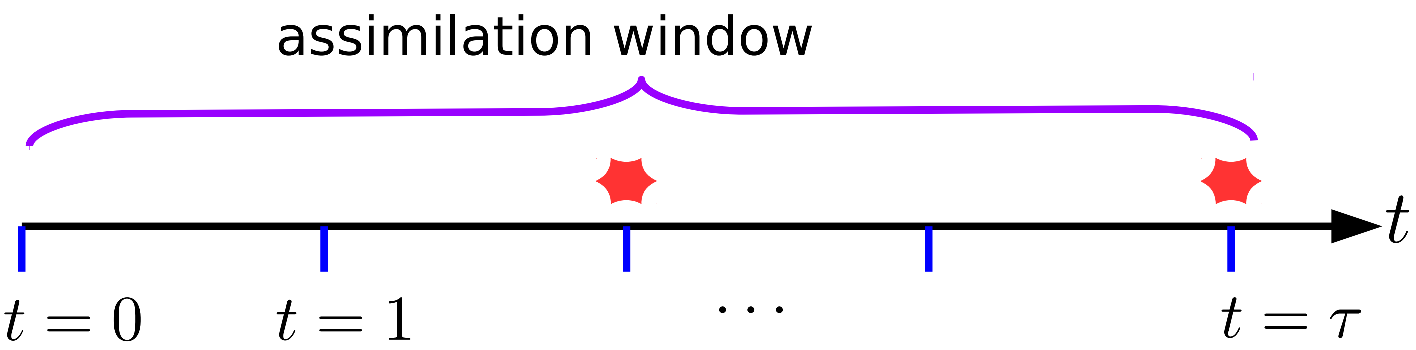


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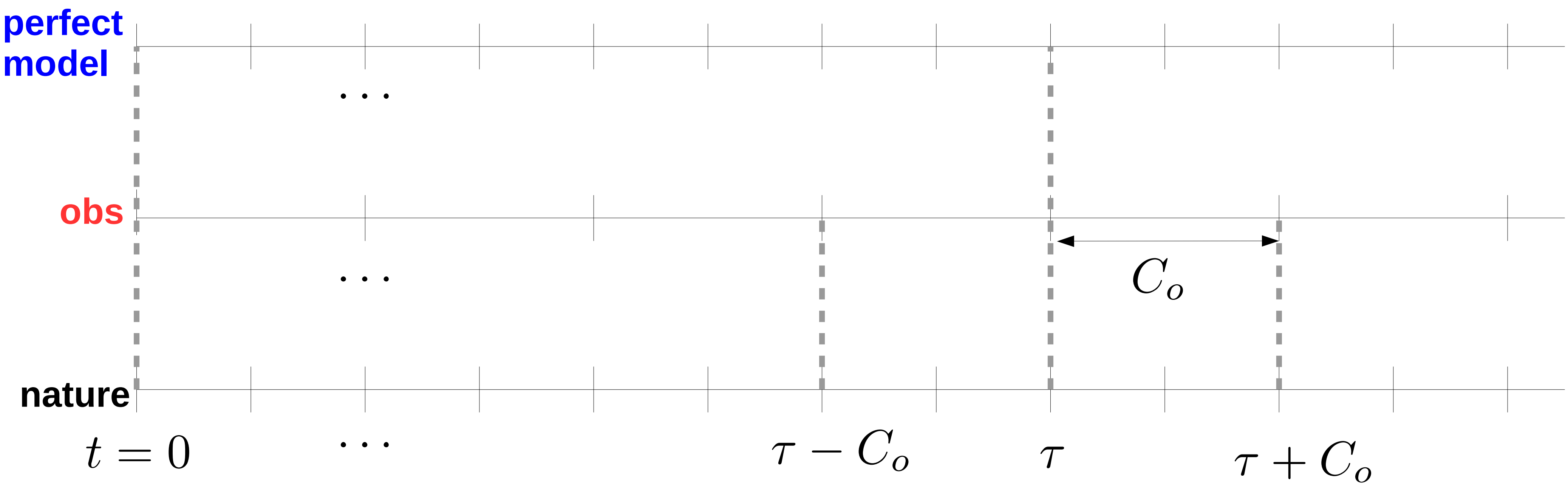
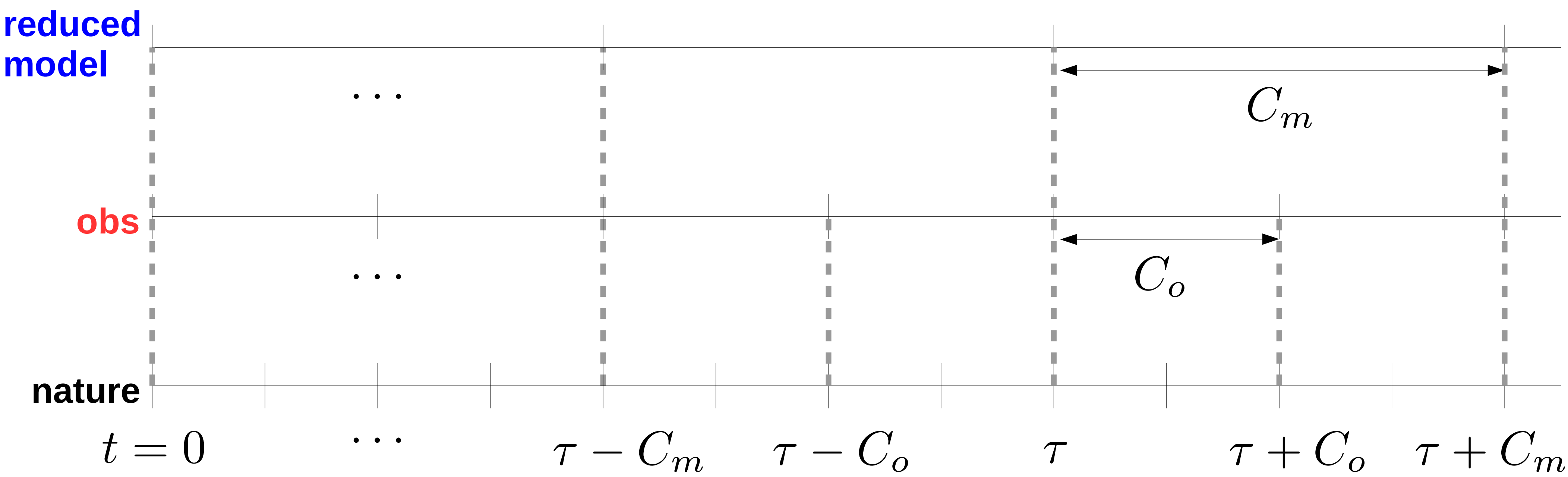
# Perfect model

(a) A **perfect model** resolving **all the temporal scales**. $\mathbf{v}^t = \mathbf{0} \; \forall \; t$



# Reduced model

(b) A **reduced model** resolving only **slow temporal scales**.




$$C_o \ll C_m$$

# Two-scale linear model

$$\mathbf{x}^{t+1} = \mathbf{M}^{t\rightarrow t+1}\mathbf{x}^{t}$$

Consider we can partition the state variable into slow and fast components:

$$\mathbf{x}^{t} = \begin{bmatrix} \mathbf{x}_s^t \\ \mathbf{x}_f^t \end{bmatrix} \qquad \mathbf{M}^{t\rightarrow t+1} = \begin{bmatrix} \mathbf{M}_{ss}^{t\rightarrow t+1} & \mathbf{M}_{sf}^{t\rightarrow t+1} \\ \mathbf{M}_{fs}^{t\rightarrow t+1} & \mathbf{M}_{ff}^{t\rightarrow t+1} \end{bmatrix}$$

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i.e. linear combinations:

$$\mathbf{x}_s^{t+1} = \mathbf{M}_{ss}^{t\to t+1}\mathbf{x}_s^t + \mathbf{M}_{sf}^{t\to t+1}\mathbf{x}_f^t$$

$$\mathbf{x}_f^{t+1} = \mathbf{M}_{fs}^{t\to t+1}\mathbf{x}_s^t + \mathbf{M}_{ff}^{t\to t+1}\mathbf{x}_f^t$$

$\mathrm{M}_{fs,sf}^{t\to t+1} = \mathbf{0}$

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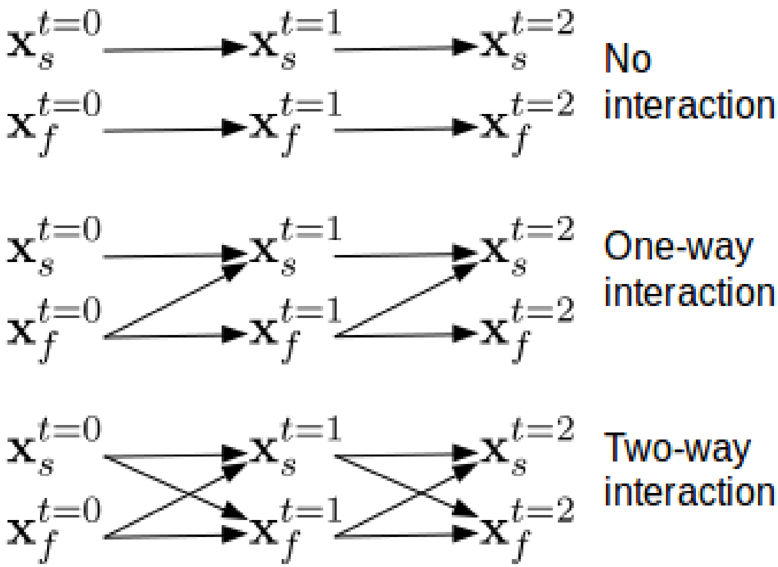


# A simple two-scale linear model

From the beginning of time (or at least the assimilation window).

$$\mathbf{x}^t = \mathbf{M}^{0\to t}\mathbf{x}^0$$

$$\mathrm{x}^t = \prod_{j=1}^{t} \mathbf{M}^{j-1\to j}\mathbf{x}^0$$

If we separate in scales:

$$\begin{bmatrix}\mathbf{x}_s^t\\ \mathbf{x}_f^t\end{bmatrix} = \begin{bmatrix}\mathbf{M}_{ss}^{0\to t} & \mathbf{M}_{fs}^{0\to t}\\ \mathbf{M}_{sf}^{0\to t} & \mathbf{M}_{ff}^{0\to t}\end{bmatrix} = \begin{bmatrix}\boxed{\mathbf{x}_s^0}\\ \mathbf{x}_f^0\end{bmatrix}$$

Actually, we do not completely know any of the matrices. But we know parts of some of them.

# Example

A simple two-scale system.

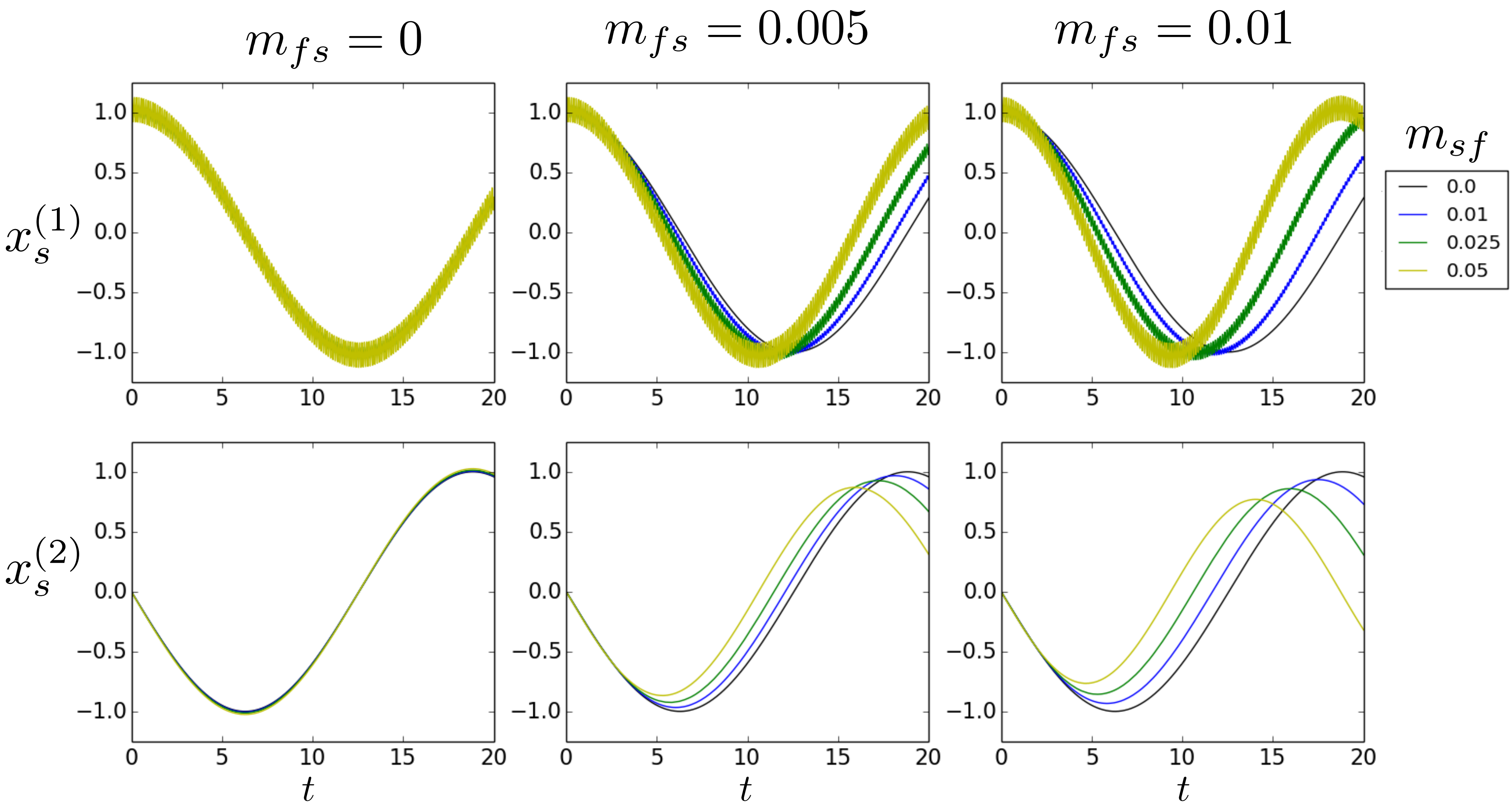
$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} \qquad \mathbf{F} = \begin{bmatrix} 0 & \omega_s & f_{sf} & 0 \\ -\omega_s & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_f \\ 0 & f_{fs} & -\omega_f & 0 \end{bmatrix} \qquad \omega_f \gg \omega_s$$

As an (autonomous) map:

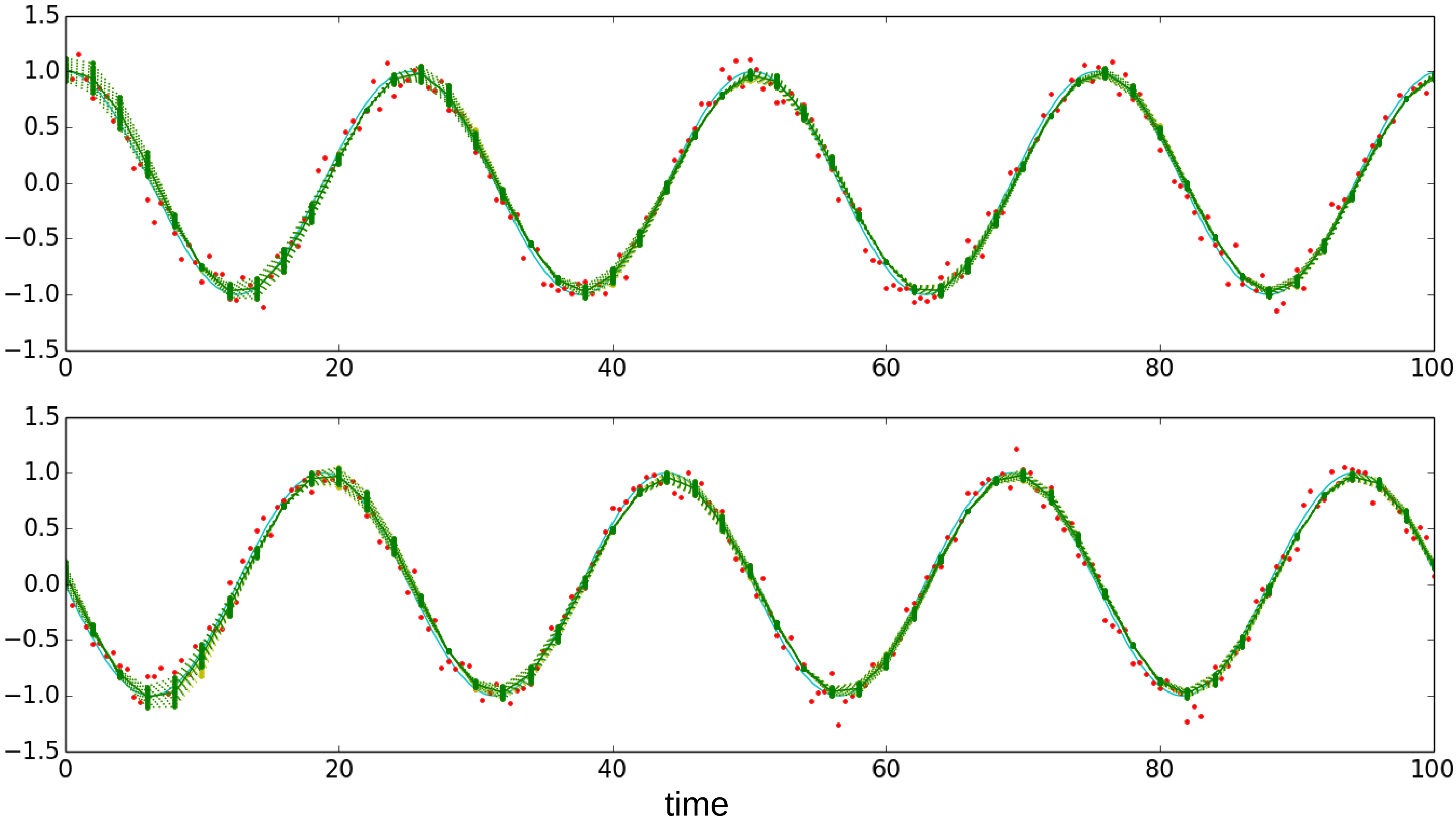
$$\mathbf{M} = \begin{bmatrix} \frac{4-\Delta^2\omega_s^4}{4+\Delta^2\omega_s^4} & \frac{4\Delta^2\omega_s^4}{4+\Delta^2\omega_s^4} & m_{sf} & 0 \\ \frac{-4\Delta^2\omega_s^4}{4+\Delta^2\omega_s^4} & \frac{4-\Delta^2\omega_s^4}{4+\Delta^2\omega_s^4} & 0 & 0 \\ 0 & 0 & \frac{4-\Delta^2\omega_f^4}{4+\Delta^2\omega_f^4} & \frac{4\Delta^2\omega_f^4}{4+\Delta^2\omega_f^4} \\ 0 & m_{fs} & \frac{-4\Delta^2\omega_f^4}{4+\Delta^2\omega_f^4} & \frac{4-\Delta^2\omega_f^4}{4+\Delta^2\omega_f^4} \end{bmatrix}$$

$$\det\,(\mathbf{M}) \approx 1 - \frac{1}{2}\left(\omega_s^2 + \omega_f^2\right)\Delta^2 + \frac{1}{8}\left(\left(\omega_s^2 + \omega_f^2\right)^2 - \underline{m_{sf}m_{fs}}\omega_f^2\omega_s^2\right)\Delta^4$$

# Evolution of slow variables

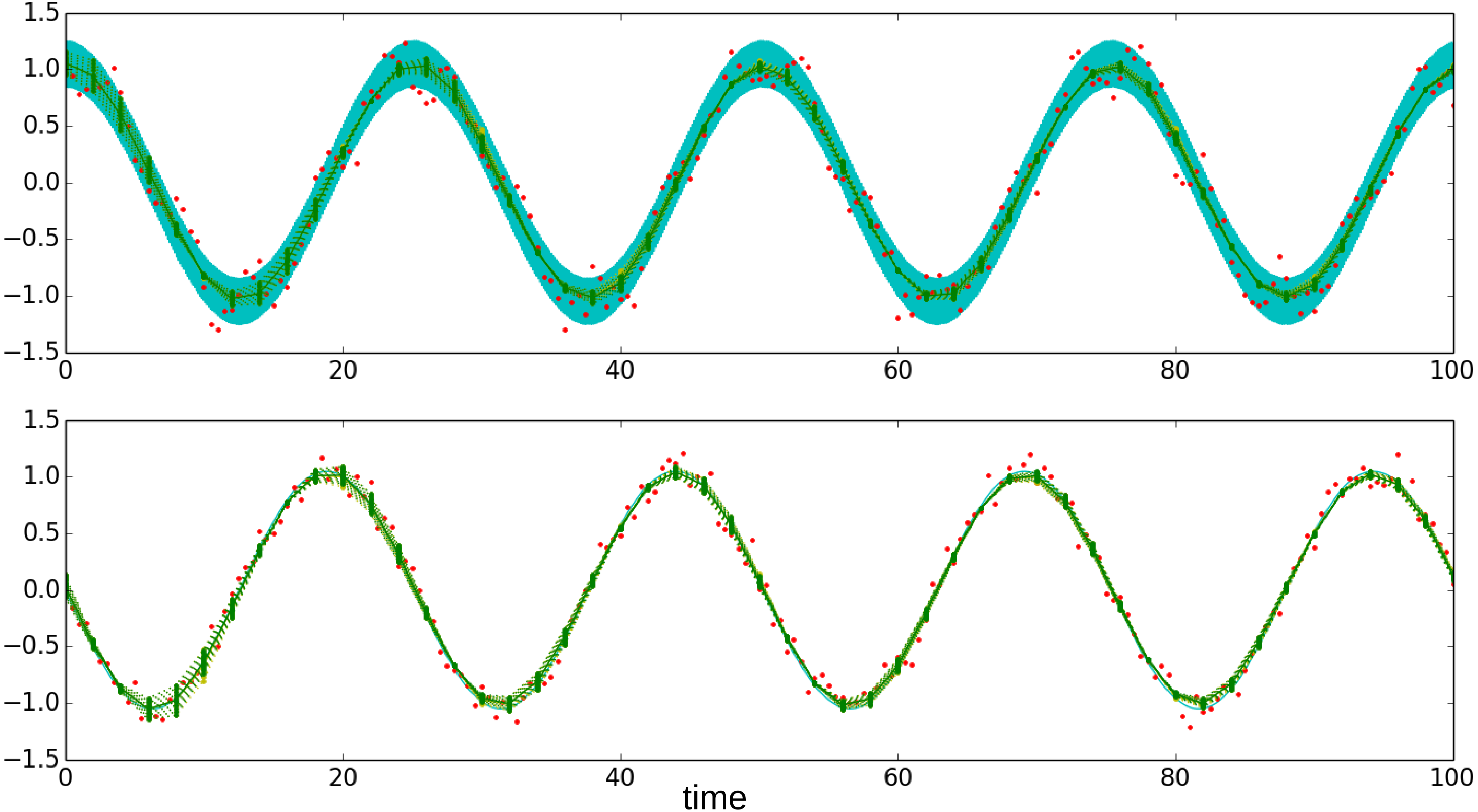


# DA in slow variables: no interactions




ETKF, 10 members, obs not every time step.

# DA in slow variables. Fast-to-slow interactions

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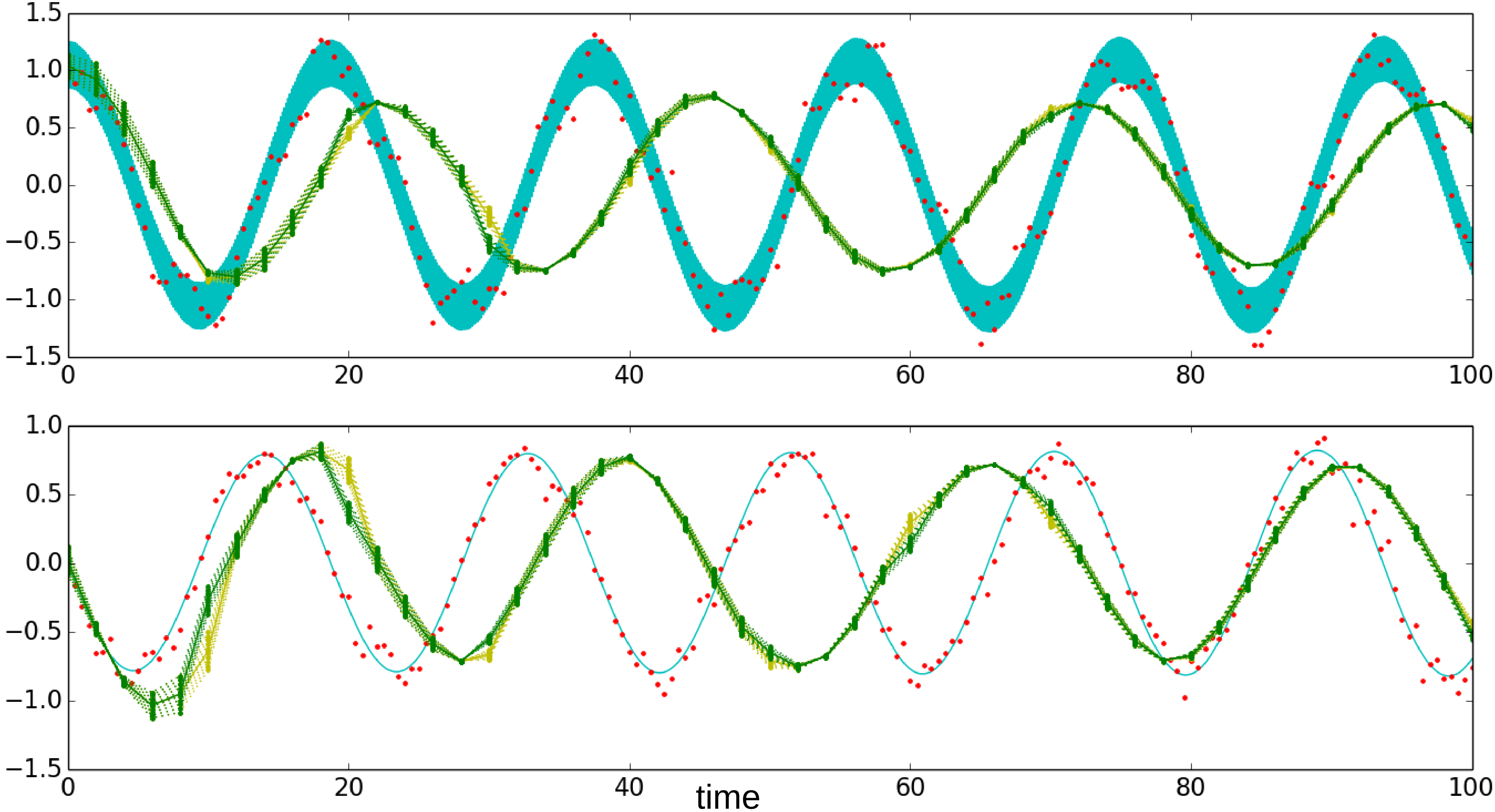
# DA in slow variables: 2-way interactions

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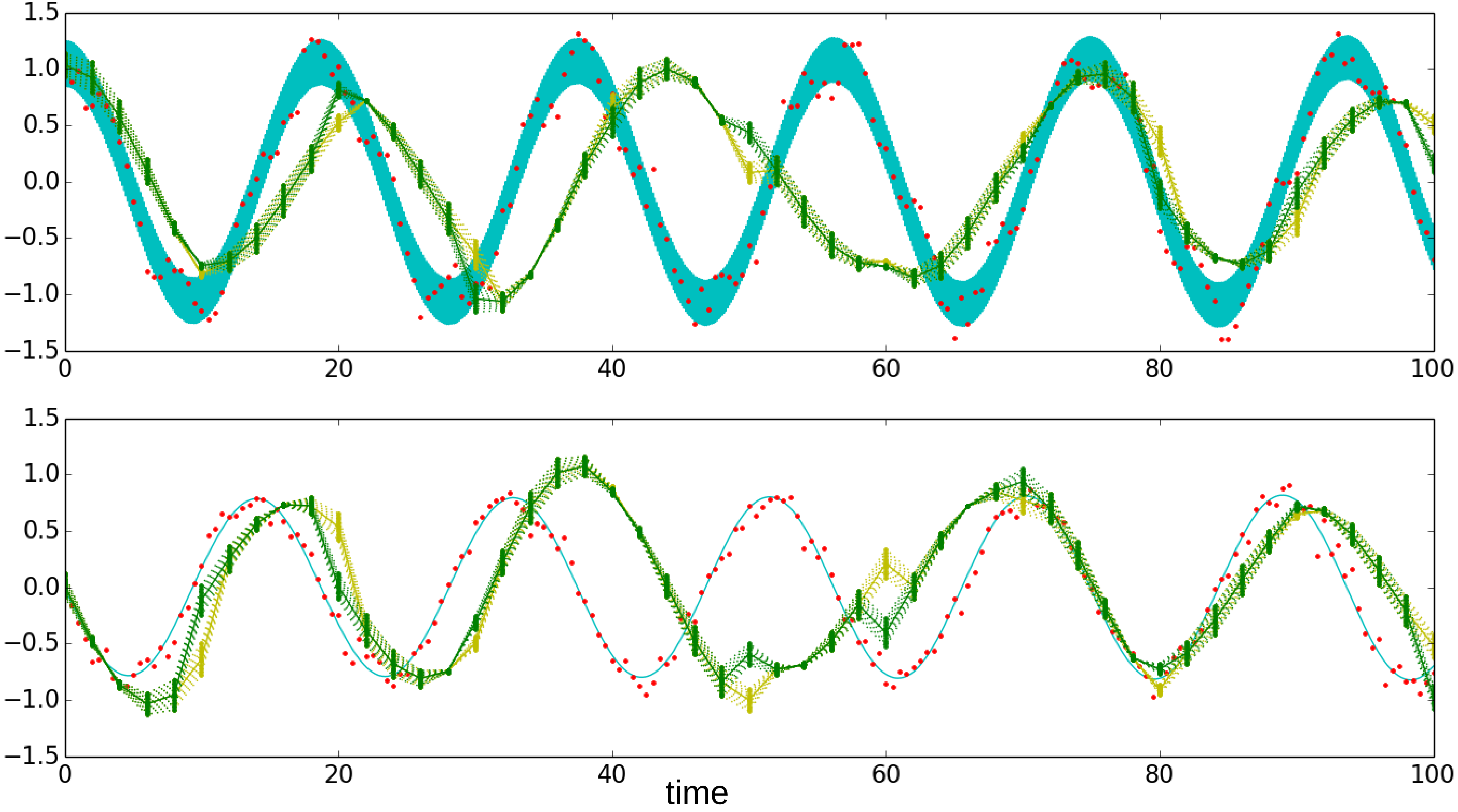
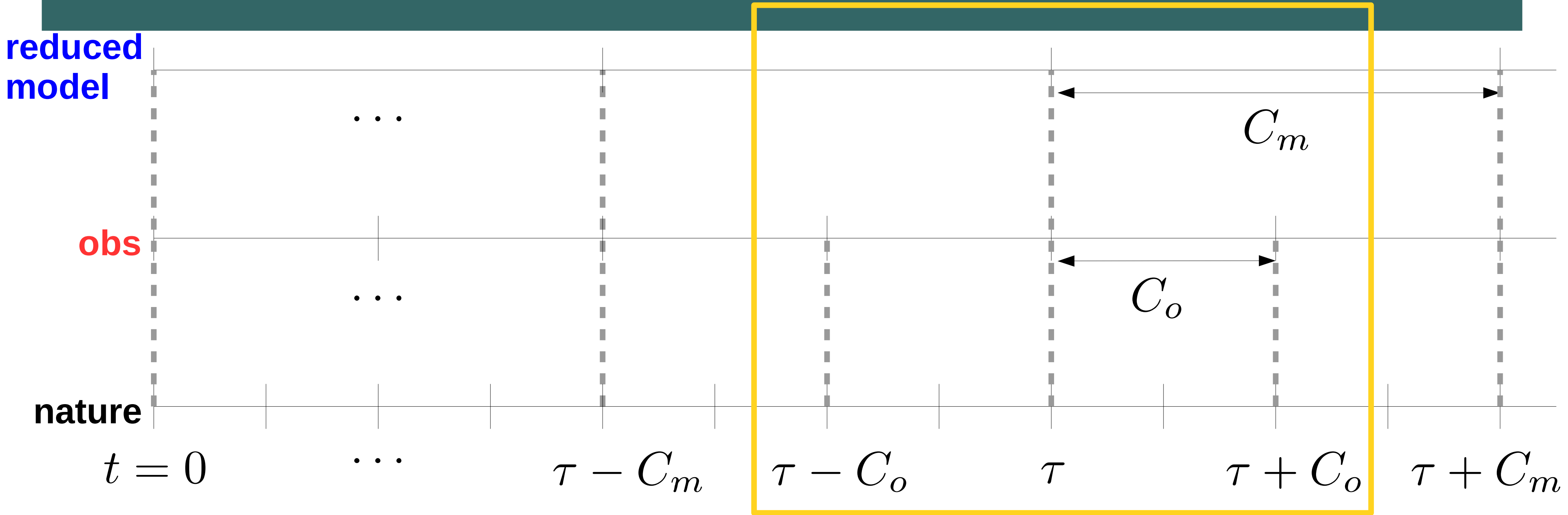
# DA in slow variables: 2-way interactions




**Inflation** was used.

# What is the DA seeing?



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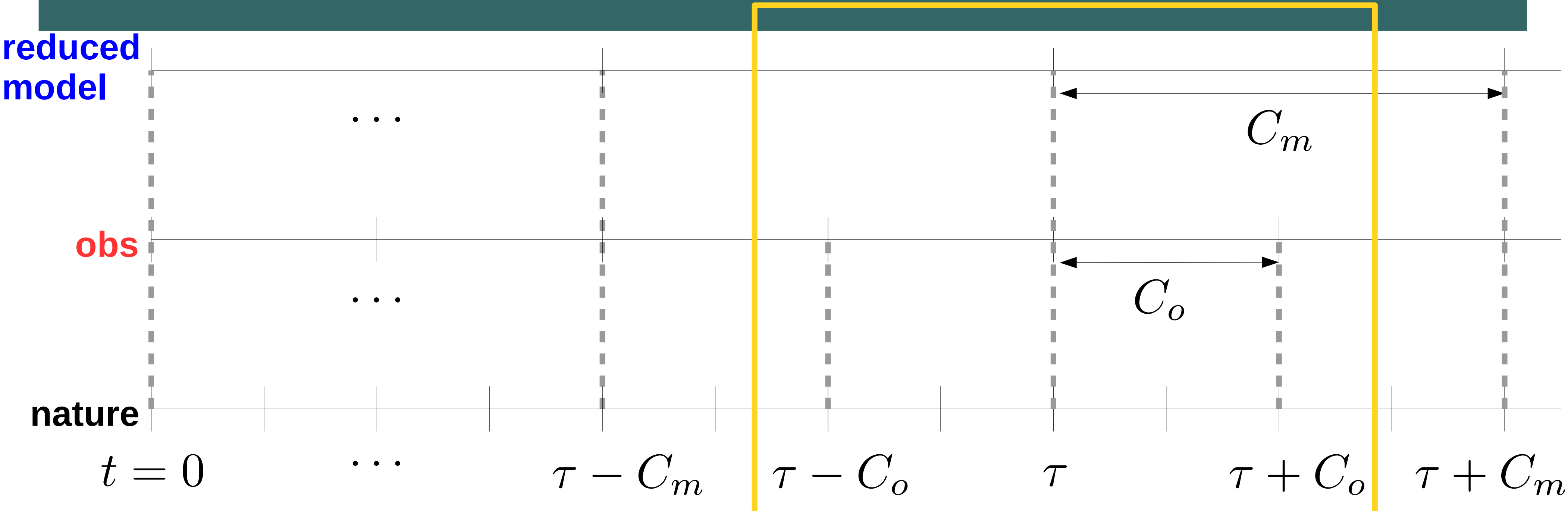


$$\underline{\mathbf{y}^{L-1}} = \mathbf{H}\left(\hat{\mathbf{x}}_s^{C_m I - C_o} + \mathbf{u}^{C_m I - C_o} + \mathbf{z}^{C_m I - C_o}\right) + \boldsymbol{\eta}^{L-1}$$

$$\underline{\mathbf{y}^{L}} = \mathbf{H}\left(\underline{\hat{\mathbf{x}}_s^{C_m I}} + \mathbf{u}^{C_m I} + \mathbf{z}^{C_m I}\right) + \boldsymbol{\eta}^{L}$$

$$\underline{\mathbf{y}^{L+1}} = \mathbf{H}\left(\hat{\mathbf{x}}_s^{C_m I + C_o} + \mathbf{u}^{C_m I + C_o} + \mathbf{z}^{C_m I + C_o}\right) + \boldsymbol{\eta}^{L+1}$$

Isolated evolution — Memory — Noise

Underlined denotes what is '**available**' either in obs or in reduced model.

# Finding the components

Let us recall:

$$\begin{bmatrix} \mathbf{x}_s^t \\ \mathbf{x}_f^t \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{ss}^{0\to t} & \mathbf{M}_{fs}^{0\to t} \\ \mathbf{M}_{sf}^{0\to t} & \mathbf{M}_{ff}^{0\to t} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_s^0 \\ \mathbf{x}_f^0 \end{bmatrix}$$

Think of three cases:

Independent $\mathrm{M}^{t\to t+1} = \begin{bmatrix} \mathbf{M}_{ss}^{t\to t+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{ff}^{t\to t+1} \end{bmatrix}$

Fast to slow $\mathrm{M}^{t\to t+1} = \begin{bmatrix} \mathbf{M}_{ss}^{t\to t+1} & \mathbf{M}_{sf}^{t\to t+1} \\ \mathbf{0} & \mathbf{M}_{ff}^{t\to t+1} \end{bmatrix}$

Two-way $\mathrm{M}^{t\to t+1} = \begin{bmatrix} \mathbf{M}_{ss}^{t\to t+1} & \mathbf{M}_{sf}^{t\to t+1} \\ \mathbf{M}_{fs}^{t\to t+1} & \mathbf{M}_{ff}^{t\to t+1} \end{bmatrix}$

# Fast-to-slow interaction

$$\mathbf{x}_s^\tau = \hat{\mathbf{x}}_s^\tau\left(\mathbf{x}_s^0\right) + \mathbf{z}^\tau\left(\mathbf{x}_f^0\right)$$

Isolated evolution

Noise

Explicitly:

$$\mathbf{x}_s^\tau = \mathbf{M}_{ss}^{0\to\tau}\mathbf{x}_s^0 + \left(\sum_{j=0}^{\tau-1}\mathbf{M}_{ss}^{j+1\to\tau}\mathbf{M}_{sf}^{j\to j+1}\mathbf{M}_{ff}^{0\to j}\right)\mathbf{x}_f^0$$

$$\mathbf{x}_f^\tau = \mathbf{M}_{ff}^{0\to\tau}\mathbf{x}_f^0$$

# Fast-to-slow interaction

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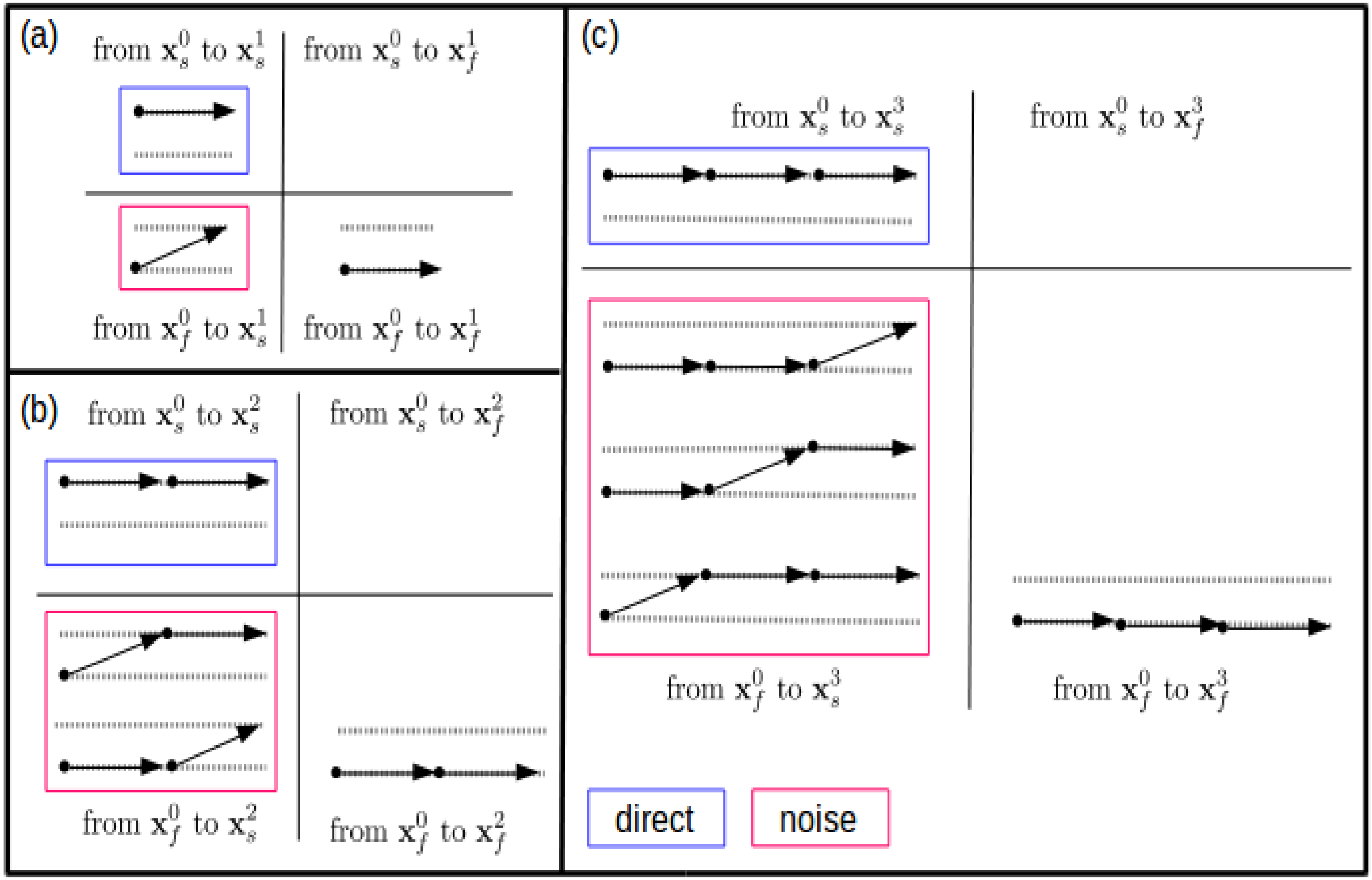

# Fast-to-slow interaction

In term of stationary (1-step transition) model errors: $\boldsymbol{\zeta}^{t+1} \sim N(\mathbf{0}, \mathbf{Q})$

$$\mathrm{x}_s^t = \mathbf{M}_{ss}^{t-1 \to t} \mathbf{x}_s^{t-1} + \boldsymbol{\zeta}^t$$

Then the effective evolution is:

$$\mathbf{x}_s^\tau = \hat{\mathbf{x}}_s^\tau \left(\mathbf{x}_s^0\right) + \mathbf{z}^\tau \left(\mathbf{x}_f^0\right)$$

$$\mathrm{z}^{\ \tau} = \sum_{t=0}^{\tau} \mathbf{M}_{ss}^{t \to \tau} \boldsymbol{\zeta}^t$$

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1-step transition model errors: $\boldsymbol{\zeta}^t = \mathbf{M}_{sf}^{t-1\rightarrow t}\mathbf{M}_{ff}^{t\rightarrow t-1}\mathbf{z}_f^0$

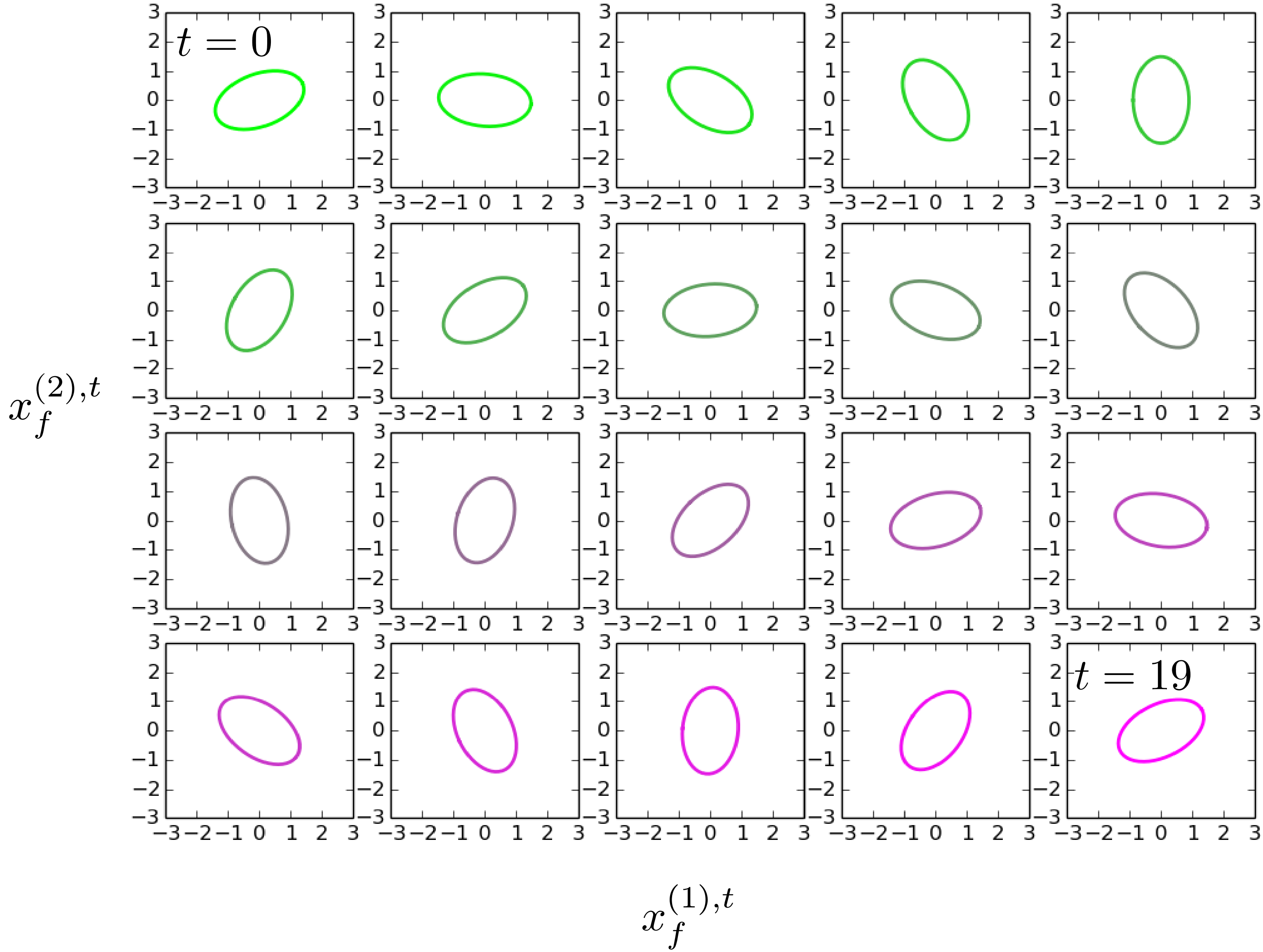
The covariance at *t* depends on the fast model. Also, the time autocorrelation of $\mathbf{z}^\tau$ depends on this.

# Fast-to-slow interaction

In our example it is right to consider the transition error stationary.

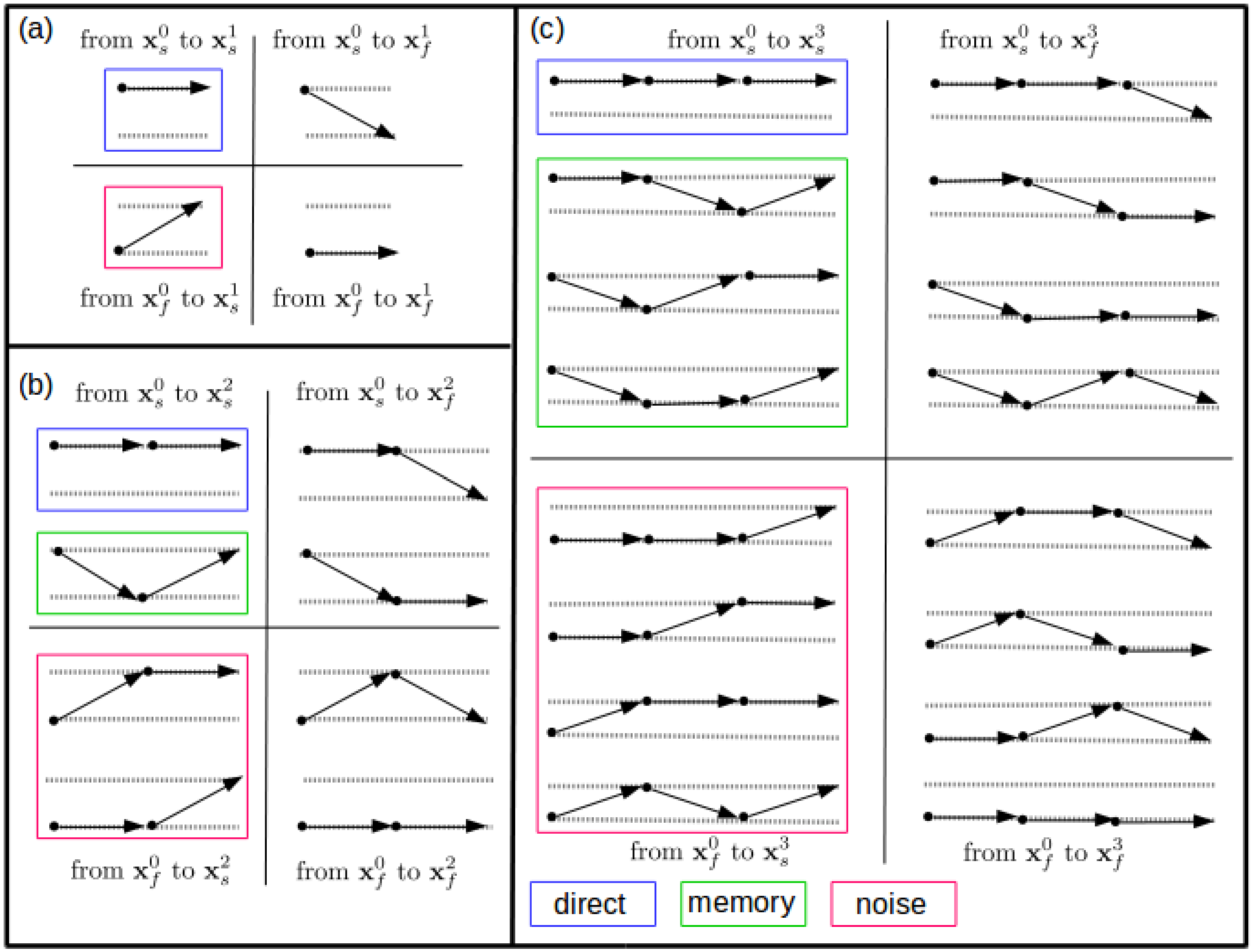
Evolution of the covariance matrix of the fast variables.

Eigenvalues conserved by the model.



# Two-way interactions

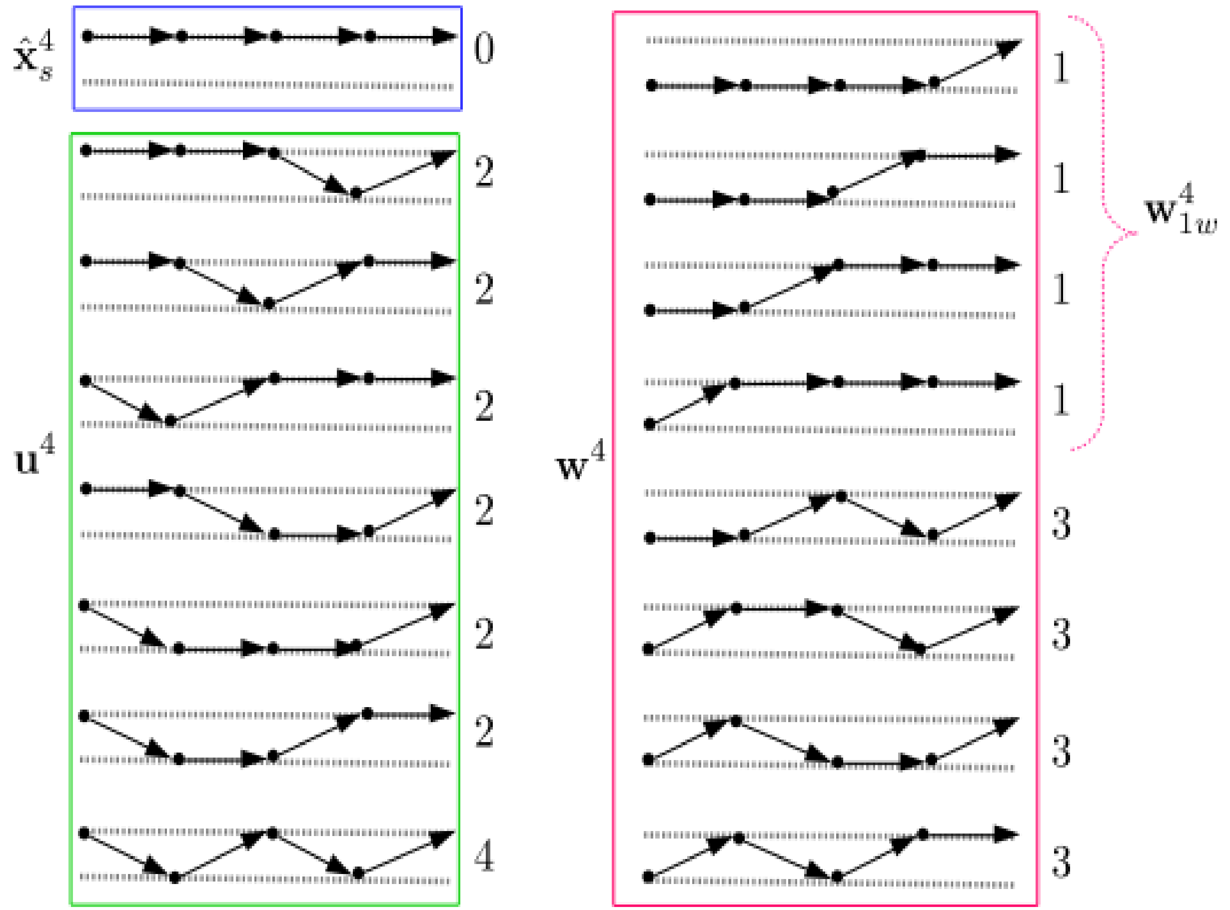
$$\mathbf{x}_s^\tau = \hat{\mathbf{x}}_s^\tau\left(\mathbf{x}_s^0\right) + \mathbf{u}^\tau\left(\mathbf{x}_s^0\right) + \mathbf{z}^\tau\left(\mathbf{x}_f^0\right)$$



# Two-way interactions

The components of the noise are not the same as those in the in the 1-way case.

$$\mathbf{x}_s^4 = \hat{\mathbf{x}}_s^4 + \mathbf{u}^4 + \mathbf{z}^4$$



# Why did we do this analysis?

$$\underline{\mathbf{y}^{L-1}} = \mathbf{H}\left(\hat{\mathbf{x}}_s^{C_m I - C_o} + \mathbf{u}^{C_m I - C_o} + \mathbf{z}^{C_m I - C_o}\right) + \boldsymbol{\eta}^{L-1}$$

$$\underline{\mathbf{y}^{L}} = \mathbf{H}\left(\underline{\hat{\mathbf{x}}_s^{C_m I}} + \mathbf{u}^{C_m I} + \mathbf{z}^{C_m I}\right) + \boldsymbol{\eta}^{L}$$

$$\underline{\mathbf{y}^{L+1}} = \mathbf{H}\left(\hat{\mathbf{x}}_s^{C_m I + C_o} + \mathbf{u}^{C_m I + C_o} + \mathbf{z}^{C_m I + C_o}\right) + \boldsymbol{\eta}^{L+1}$$

Try the following update.

$$\underline{\hat{\mathbf{x}}_s^{C_m I, b}}, \quad \mathbf{u}^{C_m I, b}, \quad \underline{\mathbf{y}^{L-1}}, \quad \underline{\mathbf{y}^{L}}, \quad \underline{\mathbf{y}^{L+1}} \quad \longrightarrow \quad \underline{\hat{\mathbf{x}}_s^{C_m I, a}}, \quad \mathbf{u}^{C_m I, a}$$

# Questions

What part of the two-way noise is completely stationary and which is not?

Can we determine statistical properties of the memory using only some paths?

How to use the extra information from neighbouring (in time) observations.

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